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1978 J. Phys. A: Math. Gen. 11 1983

(http://iopscience.iop.org/0305-4470/11/10/016)

J. Phys. A: Math. Gen., Vol. 11, No. 10, 1978. Printed in Great Britain

Percolation theory at the critical dimension

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Received 3 May 1978

Abstract. Corrections to scaling at the critical dimension have been calculated from the ϕ^3 field theory. Numerical calculations based upon series expansions for the mean cluster size in percolation theory are shown to be consistent with an asymptotic behaviour of the type found for the susceptibility in the $n=0$ limit of the ϕ^3 model.

1. Introduction

The Potts model may be defined as a classical spin model in which the n-component spin vectors $\mathbf{s}_i$ of magnitude s at each lattice site take on values corresponding to the $n+1$ corners of a hypertetrahedron (Zia and Wallace 1975). The interaction between spin pairs is via the scalar product of the spin vectors and $n=1$ corresponds to the Ising interaction.

Kasteleyn and Fortuin (1969) showed that the properties of a bond percolation model may be obtained from the thermodynamic functions of a Potts model with $s^2=n$ in the limit $n\to 0$. For example the initial susceptibility χ_0 gives the mean size of clusters S, the spontaneous magnetisation M_0 gives the percolation probability P and the pair-correlation function G_{ij} gives the pair-connectedness function P_{ij} between lattice sites i and j.

Harris *et al* (1975) argued that the critical dimension d_c for the Potts model is six on the basis that when $d=6$ the hyperscaling relation $d\nu=2\beta+\gamma$ is satisfied by the mean field exponents, $\beta=1$, $\gamma=1$, $\nu=\frac{1}{2}$. (This result is a generalisation of a previous suggestion of Toulouse (1974) for the percolation problem.) They also used renormalisation group (RG) methods to obtain the first-order term in an expansion of the exponents in powers of $\epsilon=6-d$ and this was later extended to second order by Priest and Lubensky (1976)¶ and Amit (1976). To first order the results of these calculations

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¶ The second-order terms in this paper have since been revised in agreement with Amit (1976 *Phys. Rev.* B **14** 5125).

may be written

$$
\begin{aligned}
\gamma(\epsilon) &= 1+\frac{1-n}{7-3n}\epsilon+\mathrm{O}(\epsilon^2),\\
\beta(\epsilon) &= 1-\frac{\epsilon}{7-3n}+\mathrm{O}(\epsilon^2),\\
\nu(\epsilon) &= \frac{1}{2}+\frac{5}{12}\frac{1-n}{7-3n}\epsilon+\mathrm{O}(\epsilon^2),\\
\delta(\epsilon) &= 2+\frac{2-n}{7-3n}\epsilon+\mathrm{O}(\epsilon^2).
\end{aligned}
\tag{1.1}
$$

The above results were calculated for a ϕ^3 field theory and are not necessarily correct for the true Potts model interaction. The partition function for the Potts model may be written exactly as (Zia and Wallace 1975).

$$Z=(n+1)^{|\mathscr{S}|}\Big\langle \exp\Big(\sum_{i\in\mathscr{S}} V(\boldsymbol{\phi}_i, \boldsymbol{L}_i)\Big)\Big\rangle_{\phi} \tag{1.2}$$

where the sum is over all lattice sites $\mathscr{S}$ and the average over the internal fields $\boldsymbol{\phi}_i$ is calculated with a Gaussian weight function which includes the pair interaction. The 'on-site' interaction V is given by

$$V(\boldsymbol{\phi}_i, \boldsymbol{L}_i)=\ln\langle\exp[(\boldsymbol{\phi}_i+\boldsymbol{L}_i)\cdot \boldsymbol{s}_i]\rangle_0 \tag{1.3}$$

where the average gives equal weight to all $n+1$ states. Expansion of $\ln Z$ in powers of the external fields $\mathbf{L}_i$ yields the spin correlation functions. The ϕ^3 field theory is obtained by expanding V in powers of ϕ and discarding all terms involving more than three fields. Amit *et al* (1977) have shown that although the ϕ^4 interaction makes the Gaussian fixed point unstable when $d<4$, it is irrelevant in respect of the stability of the ϕ^3 fixed point to order ϵ. Houghton *et al* (1978) have shown that for $n>1$ the ϵ expansion is not Padé–Borel summable since the coefficients, which increase factorially, all have the same sign. For $n=1$ the ϕ^3 interaction vanishes from the Hamiltonian and for $n<1$ the ϵ expansion has alternating signs and is Padé–Borel summable. These results lead one to believe that the ϕ^3 theory should lead to critical behaviour which is the same as for the percolation model ($n=0$).

In this paper comparison is made between results obtained from exact low-density series expansions for the percolation model and RG results for the ϕ^3 field theory. We work at $d=6$ where the exponents associated with the logarithmic corrections to mean field theory for the ϕ^3 model may be calculated exactly by RG methods. Our results for the percolation model are restricted to the low-density mean size where the exponent obtained compares very favourably with the RG result.

Rudnick and Nelson (1976) have shown that for the ϕ^4 field theory near $d=4$ the corrections to scaling for the susceptibility may be written

$$\chi=A\Big[1+\Big(1-\frac{B}{\epsilon\Delta_1}\Big)(t^{\epsilon\Delta_1}-1)\Big]^{\theta_\gamma}t^{-\gamma(\epsilon)} \tag{1.4}$$

where A and B are non-universal constants but Δ_1, θ_γ and γ are universal†. Δ_1 is

† $\gamma(\epsilon)$ is as defined in (1.1) for all ϵ. For convenience in the case $\epsilon<0$, we have relaxed the convention that γ is the exponent of the dominant singularity.

determined from the correction to scaling since for $\epsilon > 0$

$$\chi \sim t^{-\gamma(\epsilon)} + a t^{-\gamma(\epsilon)+\epsilon\Delta_1}. \tag{1.5}$$

Thus $\Delta_1 = \frac{1}{2}$ by comparing (1.5) with (8.6) of Brézin *et al* (1976). The form (1.4) has the merit that it describes the crossover to classical behaviour for $\epsilon < 0$. The requirement that $\chi \sim t^{-1}$ for $\epsilon < 0$ yields the result

$$\theta_\gamma = \gamma'(0)/\Delta_1 = 2\gamma'(0). \tag{1.6}$$

Now letting $\epsilon \to 0$

$$\chi \sim t^{-1}|\ln t|^{\theta_\gamma}. \tag{1.7}$$

Professor A B Harris (private communication) has extended the calculation of Rudnick and Nelson to the ϕ^3 model and verified the form (1.4). Together with Fisch (Harris and Fisch 1977) he has tested the form by exact series expansion methods for the random resistor network $n = -1$ (Fortuin and Kasteleyn 1972) where the result (1.6) is also stated.

Our result for the ϕ^3 model when $n \to 0$, obtained by solving the Callan–Symanzik equation with $d = 6$, is $\theta_\gamma = \frac{2}{7}$, whereas from the series expansion for the percolation model we obtain $\theta_\gamma = 0{\cdot}28 \pm 0{\cdot}07$. The correlation length, critical isotherm, spontaneous magnetisation and susceptibility below T_c for the ϕ^3 model are also found to have logarithmic factors consistent with (1.6) together with (1.1) The result $\Delta_1 = \frac{1}{2}$ is found also for this model by combining the results of Amit (1976) with those of Brézin *et ag* (1976).

2. Renormalisation group calculation

2.1. $H = 0, T \to T_c^+$

We take as our starting point the Callan–Symanzik equation (Brézin *et al* 1976, equation (6.40)):

$$\left[\mu\frac{\partial}{\partial\mu} + W(g)\frac{\partial}{\partial g} - \frac{1}{2}N\eta(g) - \left(\frac{1}{\nu(g)} - 2\right)t\frac{\partial}{\partial t}\right]\Gamma^{(N)}(p_i; t, g, \mu) = 0 \tag{2.1}$$

where $\Gamma^{(N)}$ is the renormalised N-point vertex function, $t = (T - T_c)/T_c$, g is a renormalised third-order interaction parameter and μ is an arbitrary momentum scale parameter. The functions W, η and ν have been found for the ϕ^3 model to second order in ϵ by Amit (1976) and setting $\epsilon = 0$ in his results

$$W(g) = wg^3 + O(g^5), \qquad \eta(g) = \tfrac{1}{5}xg^2 + O(g^4), \qquad \frac{1}{\nu(g)} - 2 = xg^2 + O(g^4), \tag{2.2}$$

where

$$w = \tfrac{1}{4}(n+1)^2(7-3n) \qquad \text{and} \qquad x = \tfrac{5}{6}(n+1)^2(n-1). \tag{2.3}$$

The calculation follows that of Brézin *et al* (1976) for the ϕ^4 model (see § VIII.B). Logarithmic corrections to scaling in six dimensions arise from the fact that $g = 0$ at the fixed point and it is the approach to zero of the functions (2.2) which determines the exponents.

Integration of (2.1) using an arbitrary scale parameter λ gives

$$\Gamma^{(N)}(p_i; t, g, \mu) = \tilde{Z}(\lambda)^{\frac{1}{2}N}\Gamma^{(N)}(p_i; t(\lambda), g(\lambda), \lambda\mu) \tag{2.4}$$

where $g(\lambda)$ is given by

$$\int_g^{g(\lambda)} \frac{dg'}{W(g')} = \ln\lambda = -\frac{1}{wg^2(\lambda)}[1+O(g^2(\lambda)\ln g(\lambda))] \tag{2.5}$$

and $\tilde{Z}(\lambda)$ and $t(\lambda)$ are determined in terms of $g(\lambda)$ by

$$\tilde{Z}(\lambda) = \exp\left(-\int_g^{g(\lambda)} \frac{\eta(g')}{W(g')}dg'\right) = \text{constant}(g(\lambda))^{-x/5w}[1+O(g^2(\lambda))] \tag{2.6}$$

and

$$t(\lambda) = t\exp\left[-\int_g^{g(\lambda)} \frac{1}{W(g')}\left(\frac{1}{\nu(g')}-2\right)dg'\right] = \text{constant}\ t(g(\lambda))^{-x/w}[1+O(g^2(\lambda))] \tag{2.7}$$

where the constants are parameters depending on g and are positive. Dimensional arguments applied to (2.4) give

$$\Gamma^{(N)}(p_i; t, g, \mu) = \tilde{Z}(\lambda)^{\frac{1}{2}N}(\lambda\mu)^D\Gamma^{(N)}\left(\frac{p_i}{\lambda\mu}; \frac{t(\lambda)}{(\lambda\mu)^2}, g(\lambda), 1\right), \tag{2.8}$$

where $D = 6-2N$.

We shall be considering $p_i = 0$ and on the left-hand side we require t to be small, but on the right-hand side we require the temperature argument to remain finite and $g(\lambda)$ to become small as $t \to 0$ so that perturbation theory may be used to evaluate $\Gamma^{(N)}$. Therefore choose λ so that

$$t(\lambda) = (\lambda\mu)^2 \tag{2.9}$$

and combining this with (2.7) gives

$$t = \text{constant}\ (\lambda\mu)^2(g(\lambda))^{x/w}[1+O(g^2(\lambda))]. \tag{2.10}$$

Equation (2.10) together with (2.5) shows that as $\lambda \to 0$ both t and $g(\lambda) \to 0$ as required. Substituting (2.6) and (2.10) into (2.8) we obtain

$$\Gamma^{(N)}(0; t, g, \mu) = \text{constant}(g(\lambda))^{-x(3-\frac{9}{10}N)/w}t^{3-N}\Gamma^{(N)}(0; 1, g(\lambda), 1)[1+O(g^2(\lambda))]. \tag{2.11}$$

In leading order perturbation

$$\Gamma^{(N)}(p_i = 0) \sim t^{3-N}g^{N-2}, \qquad N \geq 2 \tag{2.12}$$

and using this in the right-hand side of (2.11) together with (2.5), (2.9) and (2.10),

$$\Gamma^{(N)}(0; t, g, \mu) \sim t^{3-N}(w|\ln t|)^{[x(3-\frac{9}{10}N)/2w]+\frac{1}{2}(2-N)}\left[1+O\left(\frac{1}{|\ln t|}\right)\right]. \tag{2.13}$$

Since $\chi_0^{-1} = \Gamma^{(2)}(p=0)$ we find

$$\chi_0 \sim t^{-1}(w|\ln t|)^{\theta_\gamma}\left[1+O\left(\frac{1}{|\ln t|}\right)\right] \tag{2.14}$$

where $\theta_\gamma = -\frac{3}{5}x/w$. For the percolation problem $n = 0$ and hence $\theta_\gamma = \frac{2}{7}$. From (2.10) and (2.5) it follows that the correlation length has the form

$$\xi = (\lambda\mu)^{-1} \sim t^{-\frac{1}{2}}(w|\ln t|)^{\theta_\nu}\left[1+O\left(\frac{1}{|\ln t|}\right)\right] \tag{2.15}$$

where $\theta_\nu = -x/4w$ which for $n=0$ gives $\theta_\nu = \frac{5}{42}$.

2.2 *Equation of state*

So far we have considered the zero field $t>0$, region. To obtain the equation of state we start from equation (7.4) of Brézin *et al* (1976)

$$\left[\mu\frac{\partial}{\partial\mu} + W(g)\frac{\partial}{\partial g} - \frac{1}{2}\eta(g)\left(1+M\frac{\partial}{\partial M}\right) - \left(\frac{1}{\nu(g)}-2\right)t\frac{\partial}{\partial t}\right]H(M, t, g, \mu) = 0 \tag{2.16}$$

where M is the magnetisation and H the magnetic field. Integrating this equation and using dimensional arguments, the equation corresponding to equation (7.8) of Brézin *et al* (1976) is

$$H(M, t, g, \mu) = \tilde{Z}(\lambda')^{1/2}(\lambda'\mu)^4 H\left(\frac{M(\lambda')}{(\lambda'\mu)^2}, \frac{t(\lambda')}{(\lambda'\mu)^2}, g(\lambda'), 1\right) \tag{2.17}$$

where (2.5), (2.6) and (2.7) are still valid with λ replaced by λ' and $M(\lambda')$ is given by

$$M(\lambda') = M\tilde{Z}(\lambda')^{1/2}. \tag{2.18}$$

In order to use perturbation theory we require the magnetisation argument on the right-hand side of (2.17) to remain finite as $M\to 0$. Choosing λ' so that

$$M(\lambda') = (\lambda'\mu)^2 \tag{2.19}$$

and using (2.18) and (2.6) we get

$$M = \text{constant }(\lambda'\mu)^2(g(\lambda'))^{x/10w}[1+O(g^2(\lambda'))]. \tag{2.20}$$

Hence as $\lambda'\to 0$, $M\to 0$ so that this limit will give the critical behaviour. Substituting (2.19), (2.6) and (2.7) into (2.17) gives

$$H(M, t, g, \mu) = \text{constant } M^2(g(\lambda'))^{-3x/10w} H\left(1, \text{constant}\frac{t}{M}(g(\lambda'))^{-9x/10w} \times[1+O(g^2(\lambda'))], g(\lambda'), 1\right)[1+O(g^2(\lambda'))]. \tag{2.21}$$

Provided M is of order $|t|$ we can use perturbation theory to evaluate the right-hand side of (2.21) and obtain, using (2.5) and (2.20),

$$H(M, t, g, \mu) = [c_1 tM(w|\ln M|)^{3x/5w} + c_2 M^2(w|\ln M|)^{(3x/20w)-\frac{1}{2}}]\left[1+O\left(\frac{1}{|\ln M|}\right)\right], \tag{2.22}$$

where c_1 and c_2 are positive constants. Along the critical isotherm ($t=0$)

$$H \sim M^2(w|\ln M|)^{-\theta_\delta}\left[1+O\left(\frac{1}{|\ln M|}\right)\right] \tag{2.23}$$

where $\theta_\delta = \frac{1}{2} - (3x/20w)$ or $\theta_\delta = \frac{4}{7}$ for $n = 0$. The spontaneous magnetisation is given by

$$M_0 \sim (-t)[w|\ln(-t)|]^{-\theta_\beta}\left[1 + O\left(\frac{1}{|\ln(-t)|}\right)\right] \tag{2.24}$$

where $\theta_\beta = -\frac{1}{2} - (9x/20w)$ or $\theta_\beta = -\frac{2}{7}$ for $n = 0$.

Finally the zero field susceptibility for $t \to 0^-$ is obtained by differentiating (2.22):

$$\chi_0^{-1} = \frac{\partial H}{\partial M}\bigg|_{H=0}$$

$$= [c_1 t(w|\ln M_0|)^{3x/5w} + 2c_2 M_0(w|\ln M_0|)^{(3x/20w)-\frac{1}{2}}]\left[1 + O\left(\frac{1}{|\ln M_0|}\right)\right] \tag{2.25}$$

and using (2.24)

$$\chi_0 \sim (-t)^{-1}[w|\ln(-t)|]^{\theta_{\gamma'}}\left[1 + O\left(\frac{1}{|\ln(-t)|}\right)\right] \tag{2.26}$$

where $\theta_{\gamma'} = \theta_\gamma$. Notice that both terms in (2.25) have the same value of $\theta_{\gamma'}$ and we have assumed that the second term is of larger magnitude than the first in order to get a positive susceptibility.

3. Series analysis

In this section we analyse the mean size series $\bar{S}(p)$ at the critical dimension ($d = d_c = 6$) for a dominant singular structure of the form

$$\bar{S}(p) \sim (p_c - p)^{-1}|\ln(p_c - p)|^{\theta_\gamma}, \qquad p \to p_c^- \tag{3.1}$$

using a method of analysis recently devised by Guttmann (1978). The assumed asymptotic form (3.1) is of the same general type as we obtained in § 2 for the susceptibility of the ϕ^3 field theory. The mean size series $\bar{S}(p)$ for the bond problem on a general d-dimensional hypercubic lattice have recently been obtained by Gaunt and Ruskin (1978) through order p^9. In the definition of $\bar{S}(p)$, the size of a bond cluster is determined by the number of bonds it contains and not by the number of sites as it is in $S(p)$ used by Kasteleyn and Fortuin. It has been shown (Essam *et al* 1976) that $z\bar{S}(p) \geq S(p) \geq \frac{1}{2}\bar{S}(p)$, where z is the lattice coordination number, from which it follows that the precise definition used does not affect the value of p_c or the nature of the singularity, that is $S(p) \sim \bar{S}(p)$.

For $d = 6$ the critical probability was estimated to be (Gaunt and Ruskin 1978) $p_c = 0{\cdot}0941 \pm 0{\cdot}0005$. (This estimate was obtained both from exact series analysis and from a presumably asymptotic expansion for p_c in inverse powers of $\sigma = 2d - 1$.) Writing the mean size as $\bar{S}(p) = \Sigma_{n\geq 0} d_n p^n$, the series is now transformed using the transformation $u = 2p/(1 + p/p_c)$ to remove the effect of a probable singularity at or close to $p = -p_c$. The transformed series is written $\Sigma_{n\geq 0} a_n u^n$, where $u_c = p_c$ is a fixed point of the transformation. Defining the function $f(u)$ by

$$f(u) = u^{-\theta}(1-u)^{-1}|\ln(1-u)|^\theta = \sum_{n\geq 0} b_n u^n, \tag{3.2}$$

the method of analysis consists of extrapolating the quantity $R_n = (a_n/a_{n-1})/(b_n/b_{n-1})$ against $1/n$, and against $c_1/n + c_2/n^2$ for a range of values of θ. As explained by Guttmann (1978), if correction terms to the assumed asymptotic form of $\bar{S}(p)$ can be neglected, the elements of the sequence $\{R_n\}$ should remain constant and equal to $1/p_c$ at the correct value of θ. Unfortunately correction terms cannot in general be neglected, and in such cases the criteria for the selection of the optimal value of θ are firstly convergence of the linear and quadratic extrapolants defined by $S_n = nR_n - (n-1)R_{n-1}$, $T_n = \frac{1}{2}[nS_n - (n-2)S_{n-1}]$ respectively, to the assumed value of $1/p_c$, and secondly that exponent estimates and their linear extrapolants, defined by $s_n = n(R_n p_c - 1)$ and $t_n = ns_n - (n-1)s_{n-1}$, respectively, should approach zero.

Our results for three values of θ are shown in table 1. For $\theta = 0{\cdot}20$, the sequence $\{R_n\}$ is increasing towards $1/p_c$, while the quadratic extrapolants are decreasing, and are already below $1/p_c$. Exponent estimates and their linear extrapolants are approaching zero from below and above, respectively. For $\theta = 0{\cdot}25$, exactly the same qualitative behaviour is observed. However the quadratic extrapolants are closer to their expected limit point $1/p_c$, while the linear extrapolants of the exponent estimates are very close indeed to zero. Thus $\theta = 0{\cdot}25$ is favoured over $\theta = 0{\cdot}20$. For $\theta = 0{\cdot}30$ the sequence $\{T_n\}$ is still closer to its limit point $1/p_c$, but the exponent estimates are further away from zero. This value of θ is therefore equally acceptable as $\theta = 0{\cdot}25$. For $\theta = 0{\cdot}35$ (not shown in table 1) all sequences are further away from their expected limit points. From this analysis alone we could conclude that $0{\cdot}25 \leqslant \theta_\gamma \leqslant 0{\cdot}30$. However to allow for the uncertainty in the value of p_c, we widen our confidence limits and write $\theta_\gamma = 0{\cdot}28 \pm 0{\cdot}07$. This is in excellent agreement with the RG result $\theta_\gamma = \frac{2}{7} = 0{\cdot}2857\ldots$, obtained in § 2 for the $n = 0$ limit of the ϕ^3 field theory and reinforces our belief in its applicability to the bond percolation problem.

Table 1. Analysis of transformed mean size series for the $d = 6$ bond percolation problem, assuming $p_c^{-1} = 10{\cdot}627$.

θ	n	R_n	Quadratic extrapolants T_n	Exponent estimates $s_n = n(R_n p_c - 1)$	Linear extrapolants $t_n = ns_n - (n-1)s_{n-1}$
	4	10·5697	10·6871	−0·0216	0·0103
	5	10·5960	10·6431	−0·0146	0·0133
0·20	6	10·6095	10·6296	−0·0099	0·0138
	7	10·6171	10·6252	−0·0065	0·0134
	8	10·6215	10·6237	−0·0041	0·0128
	9	10·6243	10·6232	−0·0023	0·0121
	4	10·5222	10·6957	−0·0394	−0·00464
	5	10·5597	10·6486	−0·0317	−0·00058
0.25	6	10·5804	10·6334	−0·0263	0·00063
	7	10·5930	10·6281	−0·0224	0·00084
	8	10·6011	10·6260	−0·0195	0·00065
	9	10·6065	10·6250	−0·0173	0·00028
	4	10·4747	10·7046	−0·0573	−0·0198
	5	10·5233	10·6543	−0·0488	−0·0147
0·30	6	10·5512	10·6375	−0·0428	−0·0127
	7	10·5687	10·6311	−0·0384	−0·0119
	8	10·5805	10·6284	−0·0350	−0·0117
	9	10·5887	10·6269	−0·0324	−0·0117

The series for the site percolation problem (Gaunt *et al* 1976) are not so smooth and are one term shorter. Consequently the numerical evidence for site percolation is less convincing but is not inconsistent with the same value of θ_Y. At the present time there is no RG result for θ_Y for the site percolation problem.

Acknowledgments

We would like to thank Dr D J Wallace for his assistance with the renormalisation group calculations and Professors E Brézin and A B Harris for helpful correspondence.

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